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THE APPROXIMATE SYNTHESIS OF PERTURBED NON-VIBRATING SYSTEMS WITH ONE DEGREE OF FREEDOM*

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The pattern of the synthesis of a control which is optimal in speed of resp response for non-vibrating systems of a quite general form with one degree of freedom is discussed. The results of an analysis of such systems by the maximum principle /1/ are used; these results are based on constructing the switching curve of a relay control /2/. The picture of an approximate synthesis in the neighbourhood of a quiescent point (the origin of coordinates) obtained for controlled vibrating systems by asymptotic methods is complemented by the results obtained /3/.

1. Statement of the problem of synthesis that is optimal as regards speed of response for perturbed non-vibrating systems.

1.1. The initial control problem. Consider the following perturbed controlled dynamic system with one degree of freedom:

$$x^{\cdot} = y, \quad y^{\cdot} = f(x, y, u) + \varepsilon F(x, y, u) \qquad (1.1)$$
$$(x, y) \in G \subseteq R_2; \quad x(0) = x^0, \ y(0) = y^0$$

Here x, y are the system's coordinate and its velocity, i.e. the generalized phase variables, R_2 is the phase plane, a dot means differentiation with respect to time $t \in [0, T]$ $(T < \infty)$; u is a scalar control piecewise-smooth function such that $|u(t)| \leqslant 1$; $\varepsilon \in [0, \varepsilon_0]$ is a small numerical parameter $(0 < \varepsilon_0 \ll 1)$, and f, F are smooth functions of x, y and u in the domain under consideration (the perturbation function F may be continuously dependent on ε). The additional properties (smoothness, growth, etc.) of the functions f and F, and of the domain G are discussed below. It should be noted that the constraints on the control u of the form $r^-(x, y, \varepsilon) \leqslant u \leqslant r^+(x, y, \varepsilon)$ are reduced to those discussed by the linear change

$$u = {}^1/_2 (r^+ + r^-) + {}^1/_2 (r^+ - r^-) v, \quad v \in [-1, 1]$$

(where v is the new control).

For the perturbed system (1.1) we formulate the problem of defining the law of the control that is optimal regarding speed of response in the form of the synthesis of $u(x, y, \varepsilon)$ which, for sufficiently small $\varepsilon > 0$, brings the phase point $(x, y) \in G$ to the origin of coordinates (the point $(0, 0) \in G$). It is assumed that the solution of the optimal synthesis for the unperturbed problem $(\varepsilon = 0)$ is known and is in the form of a control switching curve of a relay character /1, 2/.

Below we discuss the case of non-vibrating systems (non-oscillating objects, /2/), for which the unperturbed switching curves have the simplest form: the curve consists of two semitrajectories of the unperturbed system (1.1), going to the origin and corresponding to the constant extreme values $u = \pm 1$. In /2/ the sufficient conditions are given under which the synthesis of the control $u(x, y)$, optimal regarding speed of response in the whole of the plane R_2, or in a certain open domain $G \subset R_2$ which includes the neighbourhood of the origin, and has qualitatively the same form as that for the simplest dynamic system (1.1): $x^{\cdot\cdot} = u$, $|u| \leqslant 1$. Namely, "each optimal control has no more than one switching, and the switching line passes from the second to the fourth quadrant touching the x_2-axis $(x_2 = y)$ at the origin" (see /2/).

The sufficient conditions of this picture of the optimal synthesis are as follows (see /2/). It is assumed that the function f is continuously differentiable with respect to all arguments and satisfies the monotonicity condition with respect to u

$$f_u'(x, y, u) > 0, \ (x, y) \in G, \ |u| \leqslant 1 \qquad (1.2)$$

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Further, the inequalities

$$f(0, 0, +1) = f^+ > 0, \; f(0, 0, -1) = f^- < 0 \tag{1.3}$$

hold for $u = \pm 1$ at the end point $x = y = 0$.

Conditions (1.3) secure the stopping of the system at the point indicated for a certain $u = u_0 = \text{const}$, $|u_0| < 1$ and the behaviour of the switching curve indicated above, and the optimal trajectories in the fairly close vicinity of it.

It is assumed that none of the trajectories of the unperturbed system (1.1) can go to infinity or come from it in a finite interval of time (the sufficient conditions can be in the form of the uniform Lipschitz conditions with respect to x and y, see /2/). If the motion is considered in a bounded domain G, this requirement becomes superfluous.

The following more-complicated condition (when the previous "intrinsic" conditions have been satisfied) is sufficient for the above behaviour of the optimal synthesis discussed. It is required that a function $\varphi(x, y, u)$ continuously differentiable with respect to x and y, should exist such that

$$y\varphi_x' + f\varphi_y' + \varphi^2 - \varphi f_y' - f_x' \leqslant 0, \; u = \pm 1, \; (x, y) \in G \tag{1.4}$$

It was established in /2/ by using this inequality and the maximum principle, that the optimal control $u(x, y)$ or $u(t)$ is of relay type and has no more than one switching.

1.2. Constructing an unperturbed switching curve and the pictures of the synthesis. Given $\varepsilon = 0$, the switching curve $\Pi(x, y)$ of the control $u(x, y)$ is constructed as follows. The upper left branch Π and lower right branch Π^+ of the curve are given by the relations

$$\begin{aligned} &\Pi = \Pi^- \cup \Pi^+ \\ &\Pi^\mp(x, y) = \{x, y{:}\; x^\mp(\Theta, x, y) = 0, \\ &y^\mp(\Theta, x, y) = 0, \; \Theta \geqslant 0\} \end{aligned} \tag{1.5}$$

Here Θ is the curve parameter; the functions $x^\mp(t, x^\circ, y^\circ)$, $y^\mp(t, x^\circ, y^\circ)$ are the solutions of the unperturbed Cauchy problem (1.1) for $u = \mp 1$ $(x \lessgtr 0, \; y \gtrless 0)$ respectively, which go to the origin. Under the assumptions made, these solutions can be constructed in the time interval $t \in [0, \Theta]$ during which the system's motion takes place in the domain G. The quantity Θ in (1.5) represents the time interval in which the phase point (x, y) passes to the origin along the switching curve for the fixed $u = -1$ or $u = +1$. Thus, the relations (1.5) give the parametric representation of the unperturbed switching curve $\Pi(x, y)$.

The switching curve can be written in another form, namely,

$$\Pi(x, y) = \{x, y{:}\; \Psi^\mp(x, y) = 0, \; x \lessgtr 0\} \tag{1.6}$$

where $\Psi^\mp$ are the particular solutions of the equations

$$f(x, y, \mp 1)\, dx = y\, dy, \; \Psi^\mp(0, 0) = 0$$

If we can solve the equation $\Psi^\mp(x, y) = 0$ for x, $x = x^\mp(y)$ or for y, $y = y^\mp(x)$, then the switching curve can be found in explicit form. Under the assumptions made about the smoothness of the function f in the close neighbourhood of the origin, the expression $x = {}^1/_2 f^\mp y^2 + O(y^3)$, $y \gtrless 0$ holds for the switching curve.

The curve $\Pi(x, y)$ divides the domain G into two open subdomains G^- and G^+: $G = G^- \cup G^+ \cup \Pi$. In the open subdomains and on parts from the boundaries of $\Pi^\mp$ the optimal control is $u = \mp 1$ respectively, i.e. the optimal synthesis has the form

$$\begin{aligned} &u(x, y) = -1, \\ &\forall (x, y) \in G^- \cup \Pi^- \\ &u(x, y) = +1, \\ &\forall (x, y) \in G^+ \cup \Pi^+ \end{aligned} \tag{1.7}$$

A typical pattern of behaviour of the switching curve and the trajectories for $(x, y) \in G$ (the qualitative picture of the synthesis) are shown in Fig. 1.

The Bellman function of the problem, that is the response time $T = T(x, y)$ from an arbitrary point $(x, y) \in G$ to the origin can be presented in the form $T = S + \Theta$, where S is the time of motion from the current point (x, y) to the point (ξ, η) of the intersection of the trajectory with the corresponding branch of the switching curve, and Θ is the time of motiom from the point (ξ, η) to the origin along the switching curve defined in accordance with (1.5) (see Fig. 1.) The desired quantities T, S, Θ, ξ and η can be found uniquely from the equations

$$\begin{aligned} &T(x, y) = S(x, y) + \Theta(x, y) \\ &x^\mp(S, x, y) = \xi, \; y^\mp(S, x, y) = \eta \\ &x^\pm(\Theta, \xi, \eta) = 0, \; y^\pm(\Theta, \xi, \eta) = 0 \end{aligned} \tag{1.8}$$

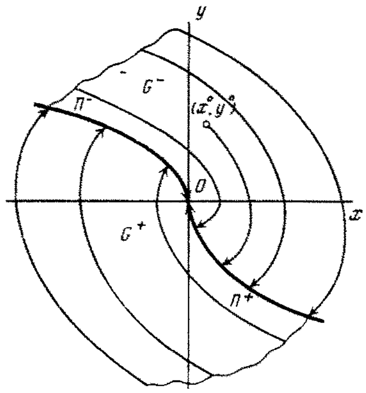

Fig. 1

The functions $x^{\mp}$ and $y^{\mp}$ are found as in (1.5); system (1.8) is reduced to two equations, in S and in Θ. Specific examples of the construction of the synthesis pattern and Bellman's functions are discussed below in Section 3.

In complex applied problems one can use the synthesis discussed above as part of the general picture of a quasi-optimal synthesis in the vicinity of the origin where, by virtue of (1.3), the control action u determines the system's motion. For example, if for large initial deflections the system's motion is oscillatory, then to quench the oscillations, that is, to bring the phase point into the neighbourhood of the origin (state of equilibrium), one can first use an approximate quasi-optimal synthesis of the simple form $u = {}^1/_2\,(u^+ + u^-) - {}^1/_2\,(u^+ - u^-)\,\mathrm{sign}\,y$, or any other which corresponds to the model of a weakly-controlled oscillation system (see /3, 4/). Then, in the domain where u is a suppressing control action, the rule (1.7) should be used for more accurate stabilization.

2. Construction of the optimal synthesis for a perturbed system. 2.1. The sufficient conditions of non-oscillation of a perturbed system. When investigating a perturbed problem of synthesis, usually all conditions, with the exception of the last condition of type (1.4), are naturally extended to the case when $\varepsilon > 0$, that is they should be satisfied uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$, and this is henceforth assumed. A serious difficulty arises with problems of the existence and construction of a smooth function $\varphi_\varepsilon(x, y, u)$ which possesses property (1.4), and the estimate of the domain $G_\varepsilon \times [0, \varepsilon_0]$ of variables x, y and ε; for this domain we similarly construct the switching curve $\Pi_\varepsilon(x, y)$ of the optimal control $u_\varepsilon(x, y)$, the optimum trajectory $x_\varepsilon(t, x^\circ, y^\circ)$, $y_\varepsilon(t, x^\circ, y^\circ)$ for $0 \leqslant t \leqslant T_\varepsilon(x^\circ, y^\circ)$ and Bellman's function $T_\varepsilon(x, y)$ of the perturbed problem. It is to be expected that for an arbitrary smooth perturbation function $F(x, y, u)$ the estimates will lead to the domain G_ε more narrow than the initial domain, i.e. $G_\varepsilon \subseteq G \subseteq R_2$ with $G_\varepsilon \to G$ as $\varepsilon \to 0$. After the problem under consideration has been solved in the domain $(x, y) \in G_\varepsilon$ the perturbed switching curve, the picture of the synthesis (optimal trajectories), and Bellman's function can, as discussed, be constructed by an analytic method (in powers of ε), or a numerical method; this does not usually present any fundamental difficulty. In Section 3 below we discuss examples of specific systems (1.1) which are of interest in practice, and for which the constructions discussed are carried out and confirmed.

For arbitrary smooth perturbations, in some cases involving the function f we may succeed in a constructive investigation regarding the existence of the function φ_ε and the estimate of G_ε.

1) For example let

$$f_x'(x, y, u) \geqslant 0, \quad u = \pm 1 \quad \forall\,(x, y) \in G \tag{2.1}$$

(see /2/).

Then for the unperturbed system $\varphi \equiv 0$, and for $\varepsilon > 0$ we can take the function $\varphi_\varepsilon = -\mu y u$, where the parameter $\mu > 0$ is to be determined. It follows from inequalities (1.3) that for sufficiently small positive μ and ε, the inequality (1.4) holds in some small neighbourhood $(x, y) \in G_\varepsilon$ of the origin. In fact, in this case $\mp \mu f(x, y, \pm 1)$, will be the main term on the left side of (1.4), which ensures the validity of this inequality since, by (1.3), we have $f(0, 0, \pm 1) = f^{\pm}$. By an appropriate choice of small parameters $\mu, \varepsilon > 0$, the remaining terms can be made fairly small in absolute value $\forall\,(x, y) \in G_\varepsilon$.

The domain G_ε will be asymptotically large with respect to the parameter ε if we can find a smooth function $\varphi(x, y, u)$ such that

$$f^2(\varphi / f)_y' + y\varphi_x' \leqslant h < 0, \quad u = \pm 1 \quad \forall\,(x, y) \in G_\varepsilon \tag{2.2}$$

Here h is a parameter; naturally, the function φ is constructed on the basis of the function f only, that is on the basis of an unperturbed system. If, however, the function f does not depend on x, then it is natural to seek the unknown φ in the form $\varphi_\varepsilon = \varphi(y, u)$, and then the inequality (2.2) will take the form $f^2(\varphi / f)_u' \leqslant h < 0$.

In particular, let $f \equiv u$; then, assuming as above that $\varphi_\varepsilon = -\mu y u$, we can arrive at the inequality

$$-\mu + \max_{(x, y) \in G_\varepsilon} [(\mu y)^2 + \varepsilon(\mu |F| + \mu |yF_y'| + |F_x'|)] \leqslant 0 \tag{2.3}$$

If the quantities $|F_y'|$ and $|F_x'|$ are uniformly bounded for $x \sim 1/\varepsilon$, $y \sim 1/\sqrt{\varepsilon}$, then the inequality (2.3) will be satisfied in an asymptotically large domain, with the above estimates of the linear dimensions. The validity of (2.3) can be established by assuming $\mu = \varepsilon M,\ M \sim 1$.

2) Let the strict inequality

$$\max_{(x,y)\in G_0} [(y\varphi - f)_x' + \varphi^2 (1 + (f / \varphi)_y')] \leqslant -\delta < 0,\quad u = \pm 1 \tag{2.4}$$

be satisfied for the known function $\varphi(x, y, u)$ in a certain bounded subdomain $G_0 \subseteq G$

Then for sufficiently small $\varepsilon > 0$ an inequality of type (1.4) will be satisfied for the perturbed system (1.1) if the initial function is $\varphi_\varepsilon = \varphi,\ \forall (x, y) \in G_0$.

However, if the domain G is unbounded, we must have additional conditions regarding the increase in the function F and its derivatives (see /2/).

For example, for the quasilinear non-vibrating system

$$x^{\cdot\cdot} + 2kx^{\cdot} + \omega^2 x = u + \varepsilon F(x, x^{\cdot}, u),\ k > \omega > 0 \tag{2.5}$$

the function φ may be taken as constant: $\varphi_\varepsilon = \varphi = -k$; then by (2.4), $\delta = k^2 - \omega^2 > 0$; the inequalities of the type (1.4) or (2.4) will be satisfied for system (2.5) in the domain $G_\varepsilon \times [0, \varepsilon_0]$, where

$$\varepsilon (kF_y' - F_x') \leqslant \delta,\ u = \pm 1 \tag{2.6}$$

If the quantities $|F_y'|$ and $|F_x'|$ are uniformly bounded for $(x, y) \in R_2$ by the constant N, then inequality (2.6) is satisfied $\forall (x, y) \in R_2$ when $\varepsilon \leqslant \delta (k + 1)^{-1} N^{-1}$.

2.2. The construction of a perturbed switching curve. In a domain G_ε uniformly bounded with respect to ε, the trajectories $x_\varepsilon^\mp(t, x^\circ, y^\circ)$, $y_\varepsilon^\mp(t, x^\circ, y^\circ)$, which enter the origin, which like (1.5), specify the desired switching curve $\Pi_\varepsilon(x, y)$ are constructed by simple recurrent procedures of successive approximations (by Picard's method /5/), or by expansions in powers of ε /6, 7/. The functions $x^\mp(t, x^\circ, y^\circ)$, $y^\mp(t, x^\circ, y^\circ)$, in Section 1 (see (1.5)) can be taken as initial approximations. The fundamental matrix which is used to construct the solutions of the subsequent approximations is obtained from $x^\mp, y^\mp$ by differentiation with respect to the parameters x°, y°. The schemes are substantiated on the basis of the Banach theorem on the compression operator /5/, or the Cauchy theorem which uses the method of Poincaré majorant functions /6/.

In an implicit form similar to (1.6), the switching curve is constructed by using particular solutions of the perturbed system (see Section 1)

$$(f + \varepsilon F)\, dx = y dy,\ u = \mp 1,\ y \gtrless 0,\ \Psi_\varepsilon^\mp (0, 0) \equiv 0 \tag{2.7}$$

The function $\Psi_\varepsilon^\mp$ can be constructed on the basis of general solutions of the unperturbed problem $\Psi^\mp(x, y) = c$ whose differentiation, on the strength of (2.7), leads to the Eqs. ($c = c^\mp$):

$$dc = \Psi_x' dx + \Psi_y' dy = -\varepsilon \Psi_x' F f^{-1} dx = \varepsilon \Psi_y' f (f + \varepsilon F)^{-1} dy\ (\Psi = \Psi^\mp,\ u = \mp 1) \tag{2.8}$$

The relation $\Psi_x^{\mp\prime} y + \Psi_y^{\mp\prime} f \equiv 0$ (with respect to x, y) is used in deriving the second and third equations in (2.8); the functions f and F are taken for $u = \mp 1$. The particular solutions of the perturbed system (2.8) $\Lambda_\varepsilon^\mp(x, y, c) = 0$, which satisfy the conditions $\Lambda_\varepsilon^\mp(0, 0, 0) \equiv 0$, together with the general solutions of the unperturbed system $\Psi^\mp = c$ yield the desired switching curve Π_ε in the form (1.6).

A substantiation of perturbation methods used to construct the switching curve Π_ε in the domain G_ε that is asymptotically large with respect to ε, requires uniform boundedness of the derivatives of the functions f and F with respect to x, y.

3. Specific mechanical systems.

3.1. A weakly perturbed dynamic system. Following Section 2, a switching curve for a non-oscillating object of the form

$$x^{\cdot\cdot} = u + \varepsilon F(x, x^{\cdot}, u),\ u^- \leqslant u \leqslant u^+,\ u^\mp \lesseqgtr 0 \tag{3.1}$$

is obtained by integrating the equation

$$dx / dy = y / u - \varepsilon F y / (u (u + \varepsilon F)),\ u = u^\mp,\ y \gtrless 0 \tag{3.2}$$

The Cauchy problem (3.2) is reduced to the non-linear integral equation in $x = x_\varepsilon(y)$:

$$x = \frac{y^2}{2u} - \frac{\varepsilon}{u} \int_0^y \frac{zF(x, z, u)}{u + \varepsilon F(x, z, u)}\, dz,\ u = u^\mp,\ y \gtrless 0 \tag{3.3}$$

The solutions of Eqs. (3.2) and (3.3) are constructed by expansions in powers of ε, or by the method of successive approximations.

The expressions $x_0^\mp = {}^1/_2\, y^2 / u^\mp,\ y \gtrless 0$ are taken as the zeroth approximation of the switching curves, that is of the functions $x_0^\mp$. The recurrent schemes will converge uniformly in the asymptotically large domain $G_\varepsilon: x \sim 1 / \varepsilon,\ y \sim 1 / \sqrt{\varepsilon}$, if the function F is differentiable with respect to x, and satisfies the Lipschitz condition.

Below we give some specific examples.

1) Let $|u| \leqslant 1$, $F = F_0 = \text{const}$; then $\varphi_\varepsilon \equiv 0$, and we arrive at the following expressions for the switching curve Π_ε and the response time T_ε when $\varepsilon |F_0| < 1\ \forall (x, y) \in G_\varepsilon = R_2$:

$$\begin{aligned} &\Pi_\varepsilon (x, y) = \{x, y\colon x = \mp {}^1/_2\, y^2 / u_*^{\mp},\ y \geqslant 0\} \\ &T_\varepsilon = S_\varepsilon + \Theta_\varepsilon,\quad S_\varepsilon = -(y + u_*^{\pm}\Theta_\varepsilon) / u_*^{\mp} \\ &\Theta_\varepsilon = [(y^2 - 2u_*^{\mp}x)/(u_*^{\pm 2} - u_*^{\pm}u_*^{\mp})]^{1/2},\quad u_*^{\mp} = 1 \mp \varepsilon F_0 \end{aligned} \tag{3.4}$$

2) If the perturbation consists of a small friction force $\varepsilon F = -\varepsilon\lambda(|y|)\,y$, where λ is a non-negative function then $\varphi_\varepsilon \equiv 0$, $\forall (x, y) \in G_\varepsilon = R_2$, $0 \leqslant \varepsilon < \infty$, and the switching curve is found from the equation

$$\Pi_\varepsilon (x, y) = \left\{x, y\colon x = \frac{y^2}{2u^{\mp}} + \frac{\varepsilon}{u^{\mp}} \int_0^y \frac{z^2\lambda(|z|)\,dz}{u^{\mp} - \varepsilon z\lambda(|z|)},\ y \gtrless 0\right\},\quad u^- \leqslant u \leqslant u^+ \tag{3.5}$$

The switching curves for a particular form of the function λ which describes the linear and quadratic friction respectively are $(y \gtrless 0)$:

$$\begin{aligned} &\varepsilon\lambda = l = \text{const},\quad x = -l^{-1}y - u^{\mp} l^{-2}\,(1 - ly / u^{\mp}) \\ &\varepsilon\lambda = \nu\,|y|,\ \nu = \text{const},\quad x = \mp {}^1/_2\, \nu^{-1} \ln (1 \mp \nu y^2 / u^{\mp}) \end{aligned} \tag{3.6}$$

The construction of a Bellman function involves the solution of perturbed transcendental equations. A corresponding generating solution is the function $T(x, y)$ defined in (3.4); for small $\varepsilon > 0$, the additions are constructed by standard perturbation methods.

3) Consider the linear perturbation $F = Ax$, $A = \text{const}$. The switching curve and the picture of the synthesis for such a system have been well studied, and constructed for arbitrary $(x, y) \in R_2$, $\alpha = \varepsilon A$ when $-u^- = u^+ = 1$. We qualitatively distinguish between oscillating and non-oscillating systems, $\alpha < 0$ and $\alpha \geqslant 0$. The switching curves and the picture of the synthesis are constructed, as discussed in Sections 1 and 2, in an asymptotically large domain G_ε $(x \sim 1/\varepsilon,\ y \sim 1/\sqrt{\varepsilon})$:

$$\begin{aligned} &\varphi_\varepsilon = -\varepsilon M y u,\ M > 0,\ u^- \leqslant u \leqslant u^+ \\ &G_\varepsilon = \{x, y\colon -Mu^2 - \alpha M x u - \varepsilon M^2 (uy) - A \leqslant 0,\ u = u^{\mp}\} \end{aligned} \tag{3.7}$$

$$\Pi_\varepsilon (x, y) = \{x, y\colon -{}^1/_2\,\alpha\,(x + u/\alpha)^2 + {}^1/_2\, u^2\alpha + {}^1/_2\, y^2 = 0,\ u = u^{\mp}, y \gtrless 0\}$$

The switching curve Π_ε in (3.7) consists of parts of ellipses when $A < 0$, and of hyperbolas when $A > 0$ (for $A = 0$ it consists of segments of parabolas; the latter case belongs to that discussed above). It should be noted that for $A \geqslant 0$ (see Section 2.1), the condition (1.4) is satisfied $\forall (x, y) \in R_2$, if $M = 0$, that is $\varphi_\varepsilon \equiv 0$.

To a first approximation in ε, we have the following explicit expression for Π_ε:

$$x = (y^2 / (2u^{\mp}))\ [1 - \alpha y^2 / (2u^{\mp})^2] - O(\alpha^2 y^4),\quad y \gtrless 0$$

As mentioned in Section 1.2, in the case of an oscillating object $(\alpha < 0)$, when x, y are large, that is $|x| \gg \Delta u / \alpha$, $|y| \gg \Delta u / \sqrt{|\alpha|}$, $\Delta u = u^+ - u^-$, to quench the oscillations we can apply a quasi-optimal synthesis of the form $u = {}^1/_2\,(u^+ + u^-) - {}^1/_2\,\Delta u\ \text{sign}\, y$ up to the values $|x| \sim \Delta u / \alpha$, $|y| \sim \Delta u / \sqrt{|\alpha|}$, and then use the control synthesis in accordance with (1.7) and (3.7).

3.2. A quasilinear non-vibrating system. The switching curve Π for the unperturbed system (2.5) can be represented in the implicit form (1.6),

$$\begin{aligned} \Pi\,(x, y) = {} &\{x, y\colon (1 - p_1 p_2 x / u + p_1 y / u)^{p_2} - \\ &(1 - p_1 p_2 x / u - p_2 y / u)^{p_1} = 0,\quad u = u^{\mp},\quad y \gtrless 0\},\quad p_{1,2} = \\ &-k \pm \sqrt{k^2 - \omega^2} \end{aligned} \tag{3.8}$$

As $\omega \to 0$, that is as $p_1 \uparrow 0$, for the switching curve $x^{\mp}(y)$ by logarithmic operation the first expression in (3.6) which corresponds to $\varepsilon\lambda = 2k$ is obtained. The representation of $\Pi(x, y)$ in the parametric form (1.5) is more convenient for further use of the perturbation method:

$$\begin{aligned} &x(\Theta) = u\omega^{-2}\,(p_1 - p_2)^{-1}\,[p_2\,(I_1(-\Theta) - 1) - \\ &\quad p_1\,(I_2(-\Theta) - 1)] = x \\ &y(\Theta) = u\,(p_1 - p_2)^{-1}\ [I_1(-\Theta) - I_2(-\Theta)] = y \\ &u = u^{\mp},\ \Theta \geqslant 0,\ I_{1,2}(\Theta) = \exp\,(p_{1,2}\,\Theta) \end{aligned} \tag{3.9}$$

In accordance with (1.8), to determine the time of optimal response $T(x, y)$, $(x, y) \in R_2$ we must solve the transcendental equation for the unknown $S = S(x, y)$, which is reduced to the form

$$\begin{aligned} &\frac{1}{p_1} \ln \frac{p_1 h_2 - h_1}{p_1 - p_2} = \frac{1}{p_2} \ln \frac{p_2 h_2 - h_1}{p_1 - p_2} \\ &h_1 = (\omega^2 / u^{\pm})\ [\sigma_1 I_1(S) + \sigma_2 I_2(S)] + (u^{\mp} / u^{\pm} - 1)\ (p_1 - p_2) \end{aligned} \tag{3.10}$$

$$h_2 = (1 / u^{\pm}) \; [\sigma_1 p_1 I_1 \; (S) + \sigma_2 p_2 I_2 \; (S)], \; p_1 p_2 = \omega^2$$
$$\sigma_1 = y - p_2 x + u^{\mp} p_2 \omega^{-2}, \; \sigma_2 = -y + p_1 x - u^{\mp} p_1 \omega^{-2}$$

If the function $S(x, y)$ is found from (3.10), then the quantities $\Theta = \Theta(x, y)$ and $T = T(x, y)$ are simply obtained:

$$T = S + \Theta, \quad \Theta = \frac{1}{p_k} \ln \frac{p_k h_2 - h_1}{p_1 - p_2}, \quad k = 1 \vee k = 2 \tag{3.11}$$

For system (2.5) when $\varepsilon > 0$ is sufficiently small, and $(x, y) \in G_\varepsilon$, where the domain G_ε is defined in accordance with (2.6), the perturbed switching curve $\Pi_\varepsilon(x, y)$ can be represented in the form

$$x_\varepsilon(\Theta) \equiv x(\Theta) + \varepsilon X_\varepsilon(\Theta) = x, \quad y_\varepsilon(\Theta) \equiv y(\Theta) + \varepsilon Y_\varepsilon(\Theta) = y \tag{3.12}$$
$$X_\varepsilon(\Theta) = [p_2 I_2(\Theta) \; I_1(-\Theta) - p_1] \; \Gamma_\varepsilon(\Theta) - p_1^{-1} I_1(\Theta) \; \Phi'(\Theta) \tag{3.13}$$
$$Y_\varepsilon(\Theta) = p_1 p_2 \; [I_2(\Theta) \; I_1(-\Theta) - 1] \; \Gamma_\varepsilon(\Theta) - I_1(\Theta) \; \Phi'(\Theta)$$
$$\Gamma_\varepsilon(\Theta) = [(p_1 - p_2) \; I_2(\Theta) + \varepsilon\Phi'(\Theta)]^{-1} \; (p_1 - p_2)^{-1}, \; u = u^{\mp}$$

(the functions $x(\Theta)$, $y(\Theta)$ have been determined in (3.9)).

The functions Φ and Φ' are determined below on the basis of a general solution of Eq.(2.5) when $u = u^{\mp}$. This solution is presented as a system of integral relations using the impulse step function $W(t)$ of an unperturbed system, i.e.

$$x(t) = c_1 I_1(t) + c_2 I_2(t) + \frac{u}{\omega^2} + \varepsilon \int_0^t W(t - \tau) F(x, y, u) \, d\tau \tag{3.14}$$
$$y(t) = c_1 p_1 I_1(t) + c_2 p_2 I_2(t) + \varepsilon \int_0^t W'(t - \tau) F(x, y, u) \, d\tau$$
$$W(t) = [I_1(t) - I_2(t)] \; (p_1 - p_2)^{-1}, \; u = u^{\mp}$$
$$\{c = (c_1, c_2) \in C_\varepsilon, \; (x, y) \in G_\varepsilon\}$$

Let the desired solutions $x(t, c, u, \varepsilon)$, $y(t, c, u, \varepsilon)$ in (3.14) be determined, for example, by Picard's method. Then

$$\Phi = \Phi(\Theta, c, u, \varepsilon) = \int_0^\Theta W(\Theta - \tau) F(x, y, u) \, d\tau \tag{3.15}$$
$$\Phi' = \partial\Phi / \partial\Theta, \; u = u^{\mp}$$

The parameters $c_{1,2}$, being functions of Θ, u and ε, are found uniquely from the quasilinear system

$$c_1 = c_2 I_2(\Theta) \; I_1(-\Theta) - \varepsilon p_1^{-1} \; I_1(-\Theta) \; \Phi' \tag{3.16}$$
$$c_2 = -p_1 \; (u\omega^{-2} + \varepsilon\Phi) \; [(p_1 - p_2) \; I_2(\Theta) - \varepsilon\Phi']^{-1}, \; u = u^{\mp}$$

The parameters $c_{1,2}(\Theta, \varepsilon)$ from (3.16) in expressions (3.15) are not differentiated with respect to Θ. System (3.16) satisfies the conditions for the existence and uniqueness of the roots $c_{1,2}$ when $\varepsilon > 0$ is sufficiently small, which are found by successive approximations or by Newton's method, /5/. On substituting the functions Φ and Φ' from (3.15) into (3.13), and then into (3.12), we obtain the desired perturbed switching curve in the parametric form

$$x = x(\Theta, u, \varepsilon), \; y = y(\Theta, u, \varepsilon), \; u = u^{\mp}, \; \Theta \geqslant 0$$

Bellman's functions of problem $T_\varepsilon(x, y)$ are constructed on the basis of the generating solution of (3.10), (3.11), by the perturbation methods discussed, using the picture of optimal synthesis for a perturbed system. Here the perturbed solution $x(t, c, u, \varepsilon)$, $y(t, c, u, \varepsilon)$ found is used.

For practical purposes, it is important to develop further the approach discussed to the study of relay control systems in which the perturbation function F also depends on time and other variables, and of multidimension systems consisting of sections of type (1.1) weakly coupled to one another, and other more general cases.

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THE JOINING OF LOCAL EXPANSIONS IN THE THEORY OF NON-LINEAR OSCILLATIONS*

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The behaviour of normal modes of oscillation in non-linear conservative systems with a finite number of degrees of freedom, when the amplitude changes from zero to infinity is studied. In the non-linear case, the normal oscillations represent a generalization of the normal oscillations of linear conservative systems (see /1/). It is assumed that the potential of a non-linear system is a polynomial of even degree in all positional variables. One can construct the trajectories of the normal oscillations in configuration space both for sufficiently small amplitude (a quasi-linear expansion), and for sufficiently large amplitude, using the fact that in these cases the system is close to a uniform system (see /2, 3/). The local expansions obtained are joined using rational-fractional Padé representations (see /4/) which enables the behaviour of oscillation modes to be followed when the amplitude changes continuously.

1. An initial conservative system is defined by the following equations of motion

$$z_i^{\cdot\cdot} + \Pi_{z_i}(z_1, z_2, \ldots, z_n) = 0 \ (i = 1, 2, \ldots, n) \tag{1.1}$$

where the potential $\Pi(z_1, z_2, \ldots, z_n)$ is a positive definite polynomial in $z_1, \ldots, z_n$ whose lowest degree is two, and the highest is $2m$. Here and below we assume that the kinetic energy is reduced to the form $T = \frac{1}{2}(z_1^{\cdot 2} + \ldots + z_n^{\cdot 2})$. An equation of this type is often encountered problems of the oscillations of non-linear elastic systems.

After separating one of the coordinates, say z_1, we use the change $z_i = cx_i$, where $c = z_1(0)$. Clearly, $x_1(0) = 1$. In addition, we can assume without loss of generality that $x_1^{\cdot}(0) = 0$. Eqs.(1.1) can be rewritten as follows:

$$x_i^{\cdot\cdot} + V_{x_i}(c, x_1, x_2, \ldots, x_n) = 0, \quad V = \sum_{k=0}^{2m-2} C^k V^{(k-2)}(x_1, x_2, \ldots, x_n) \tag{1.2}$$

where $V^{(r)}$ contains r-th degree terms with respect to the variables in the potential $V(c, x_1, x_2, \ldots, x_n) = \Pi(z_1(x_1), z_2(x_2), \ldots, z_n(x_n))$. Here the energy integral has the form

$$\sum_{i=1}^{n} x_i^{\cdot 2} + V(c, x_1, x_2, \ldots, x_n) = h \tag{1.3}$$

where h is the energy of the system. Henceforth, we shall assume that the oscillation amplitude $c = z_1(0)$ is an independent parameter, and the energy is given by (1.3). Therefore, it is convenient to represent the energy h as the sum of terms corresponding to the uniform components of the potential V,

$$h = \sum_{k=2}^{2m-2} c^k h_k \tag{1.4}$$

On introducing a new independent variable $x \equiv x_1$ and eliminating time from Eqs.(1.2) using the energy integral (1.3), we obtain equations for determining the trajectories $x_i = x_i(x)$ in the configuration space,

$$\begin{gathered} 2x_i''[h - V(c, x, x_2, \ldots, x_n)] + \Big[1 + \sum_{i=2}^{n}(x_i')^2\Big] \times \\ [-x_i' V_x(c, x, \ldots, x_n) + V_{x_i}(c, x, \ldots, x_n)] = 0 \\ (i = 2, 3, \ldots, n) \end{gathered} \tag{1.5}$$

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